Int. J. Heat Mass Transfer. Vol. 31, No. 9, pp. 1747–1757, 1988
Printed in Great Britain

0017-9310/88 \$3.00+0.00
Pergamon Press plc

Bounds for conduction and forced convection heat transfer

M. MAGEN,† B. B. MIKIC and A. T. PATERA
Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

(*Received* 24 *March* 1986 *and in final form* 18 *September* 1987)

Abstract—Bounds for heat transport in several classical problems of conduction and forced convection heat transfer are developed. The conduction bounds are based on a variational formulation, in which systematic enrichment and restriction of the space of candidate minimizing functions leads to lower and upper bounds, respectively. It is shown that the addition of insulator 'cuts' results in a broadening of the space and therefore underestimation of transport, whereas the addition of superconductor cuts leads to contraction of the space and overestimation. Our results constitute formal proof of several theorems proposed by Elrod (*Trans. ASME J. Heat Transfer* 65–70 (February 1974)). An upper bound for forced convection heat transfer in turbulent plane Couette flow based on the method of horizontal averages and power integrals is also presented. In particular, it is shown that for fixed (given) momentum transport, the heat transfer rate from the wall can be bounded from above as a function of the Reynolds and Prandtl numbers. This relationship between shear stress and heat flux serves as theoretical support for Reynolds' momentum/heat transport analogy for turbulent non-separated flow.

INTRODUCTION

A LARGE number of physical phenomena can be described by variational statements, in which the desired solution is obtained as the extremum of an appropriately defined functional [1, 2]. These variational formulations prove useful in the construction of approximate or numerical solutions, as well as in the subsequent theoretical analysis of the accuracy and convergence of these approximations [3]. In essence, the variational representation reduces the differential statement to an integral form, which is then much more amenable to estimation than the original 'pointwise' description.

It has long been known that the problem of steady heat conduction has a variational statement, in which the solution of Poisson's equation is replaced by minimization of the Dirichlet functional [4]. The variational formulation is the basis of the finite element method as applied to this class of problems [3]. However, outside of finite element discretizations, variational methods are used very little in conduction heat transfer practice, perhaps due to the non-obvious physical significance of the Dirichlet integral. In this paper, we propose to show (or, more precisely, reinvent) how simple physical considerations within the framework of a variational formulation can lead to practical yet rigorous estimation and bound techniques. Our results constitute a proof of several upper and lower bound theorems proposed by Elrod [5].

Unlike the problem of heat conduction, prediction of forced convection heat transfer does not correspond to minimization of a functional. As a result, to make progress using integral methods for convective problems requires the construction of 'artificial' functionals the extrema of which can be used to bound (but not predict) the desired quantities. Although such techniques have been employed previously in natural convection studies [2], they have not, to our knowledge, been used for forced convection. We present here an analysis of forced convection heat transfer based on these ideas. The methods and resulting bounds are much less trivial, and markedly less general than their conduction counterparts. However, they nevertheless offer insight into the basic question of the relationship between momentum and heat transport.

In Part 1 of this paper we discuss and prove some simple variational conduction bounds. Emphasis is both on the formal proofs of lower and upper bounds, and on the physical significance of the estimation techniques. Having introduced the concept of bounds, we turn in Part 2 to estimation of an upper bound for forced convection heat transfer in turbulent plane Couette flow. The relevance of these convective transport estimates to classical momentum/heat transport relations such as the Reynolds analogy [6] is discussed.

† Current address: Department of Applied Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel.

PART 1. CONDUCTION

Problem statement

We consider here the simple but common case of steady (two-dimensional) conduction heat transfer between two isothermal surfaces, as shown in Fig. 1. The governing equations and associated boundary conditions in non-dimensional form are

$$\nabla^2\theta = 0 \qquad \text{in } D \tag{1a}$$

$$\nabla\theta\cdot\mathbf{n} = 0 \qquad \text{on } \partial D_2 \tag{1b}$$

NOMENCLATURE

A	area of region B defined in the Appendix
$A(x)$	cross-section of wedge in Fig. 4
B	two-dimensional region defined in the Appendix
$B(y)$	'length' for geometry of Fig. 5
D	two-dimensional region defined in Part 1
∂D	boundary of region D
$2d$	distance between two parallel plates in Part 2
$\bar{F}, f$	dimensional and non-dimensional mean momentum flux
h	heat transfer coefficient
h'	non-dimensional heat transfer coefficient
$I(v)$	Dirichlet integral of function v, defined in equation (4b)
$\tilde{I}(v)$	modified Dirichlet integral defined in equation (31b)
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the x-, y-, z-directions
k	thermal conductivity
$\mathbf{n}, \mathbf{t}$	unit vectors orthogonal and tangential to boundaries
P, p_0, p	general, mean, and fluctuation pressure fields
Pe	Peclet number
Pr	Prandtl number
Q, q	dimensional and non-dimensional heat transfer rate
Re	Reynolds number
T	dimensional temperature
ΔT	temperature difference between isothermal boundaries
$u_0\mathbf{i}$	mean velocity field
u, v, w	fluctuation velocity fields in the x-, y-, z-directions
$\mathbf{V}, \mathbf{v}$	general and fluctuation vector velocity fields
x, y, z	Cartesian coordinates.

Greek symbols

α	thermal diffusivity of the fluid
δ	small parameter defined in the Appendix
Θ, θ_0, θ	general, mean, and fluctuation temperature fields
θ	non-dimensional temperature field in Part 1
μ	viscosity
ν	kinematic viscosity
ρ	constant density of fluid.

Other symbols

$\mathscr{B}$	space of functions defined in equation (6)
$\mathscr{B}^I$	space of functions defined in equation (13b)
$\mathscr{B}^S$	space of functions defined in equation (17b)
$\mathscr{F}$	any space of functions
$\mathscr{H}^1$	space of functions defined in equation (5)
$\langle \cdot \rangle$	integration over finite domain D (or D', D'') in Part 1
$\langle \cdot \rangle$	average over entire infinite domain in Part 2
$\bar{v}$	average over infinite horizontal plane for any function $v(x, y, z)$ in Part 2
∇	vector gradient operator.

$$\theta = 1(0) \quad \text{on } \partial D_1(\partial D_0) \tag{1c}$$

where $\theta = (T-T_0)/(T_1-T_0)$ is the non-dimensional temperature, D the physical domain, ∂D_2 the adiabatic boundary (outward normal $\mathbf{n}$), and $\partial D_1, \partial D_0$ the isothermal surfaces.

Of interest is determining the heat transfer rate through the body, which is given by

$$q = Q/k\Delta T = \int_{\partial D_1} \nabla\theta \cdot \mathbf{n}\, ds = -\int_{\partial D_0} \nabla\theta \cdot \mathbf{n}\, ds \tag{2}$$

where Q is the dimensional heat transfer rate per unit depth into the paper, $\Delta T = T_1 - T_0$ the temperature difference between the plates, and k the (assumed constant) thermal conductivity of the material. Equality of the two integrals in equation (2) follows by integration of equation (1a) over the domain, and application of Gauss' theorem and boundary conditions (1b).

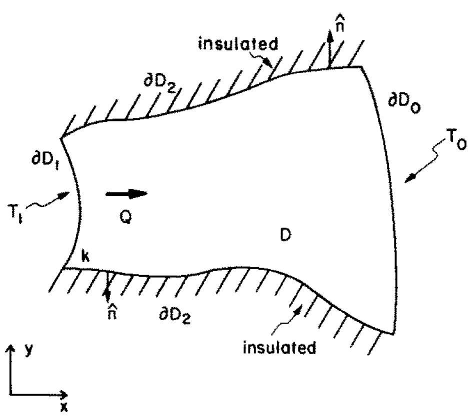

FIG. 1. General conduction problem to be considered, corresponding to heat transfer between two isothermal surfaces.

To put equation (2) in more convenient form for purposes of estimation, we multiply equation (1a) by the solution, θ, and perform an integration by parts

$$\langle \theta \nabla^2 \theta \rangle = \int_{\partial D_1 \cup \partial D_0} \theta \nabla \theta \cdot \mathbf{n} \, \mathrm{d}s - \langle (\nabla \theta)^2 \rangle = 0 \quad (3)$$

(where $\langle \cdot \rangle$ represents integration over the domain D), from which it follows by definition of q in equation (2) that

$$q = I(\theta) \quad (4a)$$

where $I(v)$ is the Dirichlet integral given by

$$I(v) = \langle (\nabla v)^2 \rangle. \quad (4b)$$

In addition to the differential statement given in equation (1), a variational formulation of the heat conduction problem can also be posed. To arrive at the variational statement, we first define the 'basic' space of functions, $\mathscr{H}^1$, for which the Dirichlet integral defined in equation (4b) makes sense

$$\mathscr{H}^1 = \{v(x,y) \mid x, y \in D, \langle v^2 \rangle < \infty, \ \langle (\nabla v)^2 \rangle < \infty\}. \quad (5)$$

We next introduce a 'subspace' of $\mathscr{H}^1$, $\mathscr{B}$, which includes only those functions from $\mathscr{H}^1$ which satisfy the essential boundary conditions on ∂D_0 and ∂D_1

$$\mathscr{B} = \{v(x,y) \mid v \in \mathscr{H}^1 \text{ in } D, \ \ v = 1(0) \text{ on } \partial D_1(\partial D_0)\}. \quad (6)$$

Note $v \in \mathscr{B}$ need not satisfy the natural boundary conditions on ∂D_2, as these conditions will be taken care of 'automatically' by the variational statement. We also assume here that all the boundaries ∂D are piecewise smooth.

Armed with the spaces described above, we can now present the variational statement associated with equation (1): find that function $\theta'(x,y)$ in $\mathscr{B}$ which minimizes the value of the Dirichlet functional I

$$I(\theta') = \inf_{v \in \mathscr{B}} I(v) \quad (7)$$

(for practical purposes 'inf' can be read as 'min'). It can be shown that the differential, (1), and variational, (7), formulations of the heat conduction problem are closely related. Indeed, from equating to zero the first variation of the functional $I(\theta')$, it follows that for sufficiently smooth data

$$\theta(x,y) \equiv \theta'(x,y). \quad (8)$$

That is, the function θ' in $\mathscr{B}$ which minimizes the functional I is the solution to the differential equation (1), θ. As regards the class of admissible v in equation (7), it is critical to note that the variational formulation (7) requires of its candidates only square integrability of *first* derivatives (i.e. the function must be continuous, but its first derivative need not be) and satisfaction of *essential* boundary conditions. In contrast, the differential statement (1) makes sense only for functions for which we assume existence (square integrability) of the *second* derivative and satisfaction of both *essential* and *natural* boundary conditions.

From equations (4), (7) and (8) we now obtain the following expression for the heat transfer rate:

$$q = \inf_{v \in \mathscr{B}} \langle (\nabla v)^2 \rangle \quad (9)$$

which will serve as the basis for the constructive bounds presented below.

Lower and upper bounds

From equation (9) it is now simple to see how lower and upper bounds can be constructed. If we define

$$q_{\mathscr{F}} = \inf_{v \in \mathscr{F}} \langle (\nabla v)^2 \rangle \quad (10)$$

for some space of functions $\mathscr{F}$, it then follows that

$$q_{\mathscr{B}^{\mathrm{LB}}} \leqslant q \leqslant q_{\mathscr{B}^{\mathrm{UB}}} \quad (11a)$$

for

$$\mathscr{B}^{\mathrm{UB}} \subset \mathscr{B} \subset \mathscr{B}^{\mathrm{LB}}. \quad (11b)$$

Enrichment of the space ($\mathscr{B} \subset \mathscr{B}^{\mathrm{LB}}$) leads to lower bounds (LB), while restriction of the space ($\mathscr{B}^{\mathrm{UB}} \subset \mathscr{B}$) leads to upper bounds (UB). This result is, of course, as old as the variational formulation of the conduction problem. Of interest here are two 'spaces' which have a simple physical interpretation and to which expressions (11) then add proof.

In what follows, the functional spaces introduced will be defined in terms of 'cuts' inserted into the original domain, D. Although these cuts are important in that they define the continuity requirements on admissible functions, they do not actually affect the domain of integration in equation (10), as they are of measure zero. We therefore do not distinguish between various forms of equation (10) defined on domains differing only by cuts, and keep the same symbol for the Dirichlet integral over all such regions.

Lower bounds—insulators

Consider the new problem in which we take the original problem defined in Fig. 1, and insert anywhere in the domain a smooth *insulator* cut, as shown in Fig. 2. The problem in differential form is then

$$\nabla^2 \theta = 0 \quad \text{in } D' \quad (12a)$$

$$\nabla \theta \cdot \mathbf{n} = 0 \quad \text{on } \partial D_2' \quad (12b)$$

$$\theta = 1(0) \quad \text{on } \partial D_1'(\partial D_0') \quad (12c)$$

$$\nabla \theta \cdot \mathbf{n} = 0 \quad \text{on } \partial D_I' \quad (12d)$$

where D' is identical to the original domain D save for a 'cut' corresponding to the insulator.

The expression for q (even for modified problem (12)) is again given by the Dirichlet integral over the domain D', as the additional boundary terms that would be generated in equation (3) all vanish due to the zero flux conditions (12d). In variational form, the heat transfer rate is thus given by

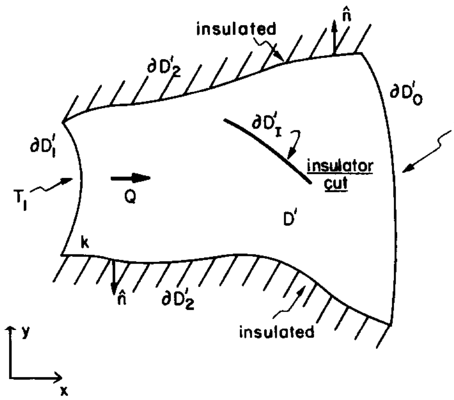

FIG. 2. Definition of lower bound problem corresponding to introduction of insulator cuts.

$$q^I = \inf_{v \in \mathscr{B}^I} \langle (\nabla v)^2 \rangle \tag{13a}$$

where $\mathscr{B}^I$ is the space of functions defined by

$$\mathscr{B}^I = \{v \mid v \in \mathscr{H}^1 \text{ in } D'; \quad v = 1(0) \text{ on } \partial D_1(\partial D_0)\}. \tag{13b}$$

Although equations (13) and (9) may appear identical, this is not the case due to the fact that D and D' are not the same. In particular, a function in $\mathscr{B}^I$ may be discontinuous across ∂D_I in D', whereas a function in $\mathscr{B}$ may not be discontinuous across the corresponding internal line in D. This implies that $\mathscr{B} \subset \mathscr{B}^I$, and that therefore from expression (11)

$$q^I \leqslant q. \tag{14}$$

The conclusion is thus that in all circumstances addition of (any number of) insulation cuts will lower the heat transfer rate through the body—as is expected on physical grounds. This general result was first stated and applied by Elrod [5], however, the demonstration of inequality (14) given in ref. [1] involves the introduction of artificial internal heat transfer coefficients, and is both significantly more complicated and less rigorous than the simple variational arguments given here.

Although it is difficult to *a priori* estimate the magnitude of the underestimation for arbitrary cuts, it is simple to understand the origin of the error. In particular, it is clear that if the insulator cut is chosen so as to be coincident with a flux line of the exact solution (e.g. parallel to $\nabla\theta$), the insulator solution will be exact, $q^I = q$. It therefore follows that the closer the insulator cut approximates a flux line, the better (higher) the lower bound will be.

Upper bounds—superconductors

Consider now another new problem in which we take the original problem of Fig. 1 and insert anywhere in the domain a *superconductor* cut, as shown in Fig. 3. The problem in differential form is then given by

$$\nabla^2\theta = 0 \qquad \text{in } D'' \tag{15a}$$

$$\nabla\theta \cdot \mathbf{n} = 0 \qquad \text{on } \partial D_2'' \tag{15b}$$

$$\theta = 1(0) \qquad \text{on } \partial D_1''(\partial D_0'') \tag{15c}$$

$$\nabla\theta \cdot \mathbf{t} = 0 \qquad \text{on } \partial D_S'' \tag{15d.1}$$

$$\int_{\partial D''_{S+}} \nabla\theta \cdot \mathbf{n}\, \mathrm{d}s = -\int_{\partial D''_{S-}} \nabla\theta \cdot \mathbf{n}\, \mathrm{d}s \tag{15d.2}$$

where $\partial D''_{S+}$ and $\partial D''_{S-}$ are the two sides of the superconductor cut, with normal and tangential vectors denoted $\mathbf{n}$ and $\mathbf{t}$, respectively. It should be noted that although equations (15d) are 'non-standard' boundary conditions for Poisson's equation (i.e. not Dirichlet, Neumann, or mixed), they can be shown to result in a well-posed elliptic problem [7]. Physically, superconductor boundary conditions (15d) correspond to the fact that the temperature along a superconductor is constant but unknown (15d.1), with the indeterminacy being fixed by condition (15d.2) which represents an energy balance on an infinitesimal control volume surrounding the cut.

The expression for the heat transfer rate, q, for the superconductor problem is the same as that given previously in equation (10). To show this, we start again with integration by parts

$$\langle \theta\nabla^2\theta \rangle = q + \int_{\partial D''_S} \theta\nabla\theta \cdot \mathbf{n}\, \mathrm{d}s - \langle (\nabla\theta)^2 \rangle = 0 \tag{16}$$

where $\partial D''_S$ refers to integration over both sides of the superconductor cut. As the surface integral over $\partial D''_S$ vanishes due to the *combination* of equations (15d.1) and (15d.2), the heat transfer rate in variational form can be written as

$$q^S = \inf_{v \in \mathscr{B}^S} \langle (\nabla v)^2 \rangle \tag{17a}$$

where $\mathscr{B}^S$ is the space of functions defined by

$$\mathscr{B}^S = \{v \mid v \in \mathscr{H}^1 \text{ in } D'';$$
$$v = 1(0) \text{ on } \partial D_1(\partial D_0), \quad \nabla v \cdot \mathbf{t} = 0 \text{ on } \partial D_S\}. \tag{17b}$$

Note equations (15d) are mixed essential/natural

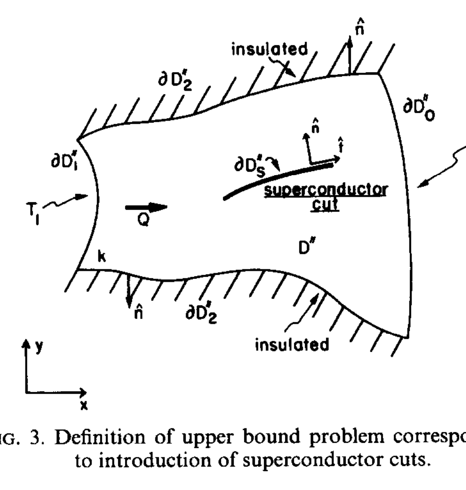

FIG. 3. Definition of upper bound problem corresponding to introduction of superconductor cuts.

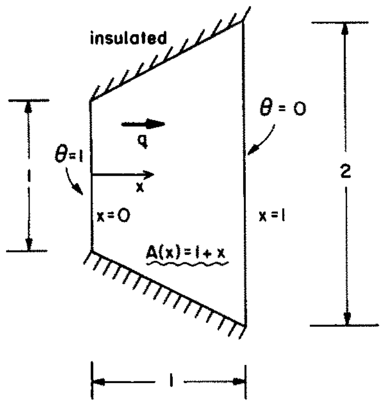

FIG. 4. Wedge geometry used to illustrate the one-dimensional estimation techniques for upper bounds.

boundary conditions, in that equation (15d.1) is essential whereas equation (15d.2) is natural.

As expression (17a) is identical in form to that given in equation (10), our general arguments as to lower and upper bounds given in expressions (11) apply. In particular, inspection of equation (17b) indicates that the two spaces $\mathscr{B}$ and $\mathscr{B}^S$ are the same except for the restrictions they place on v along $\partial D_S''$; in $\mathscr{B}^S$ functions v must be continuous and constant, whereas in $\mathscr{B}$ functions v need only be continuous. Thus, $\mathscr{B}$ is 'richer' than $\mathscr{B}^S$ ($\mathscr{B}^S \subset \mathscr{B}$), from which it then follows that

$$q \leqslant q^S. \tag{18}$$

The addition of (any number of) superconductor cuts anywhere in the domain increases the heat transfer rate—as is expected on physical grounds [5]. As in the case of insulators where the error in estimation is 'proportional' to the deviation in the insulator cut from a *flux line* of the exact solution, so in the case of superconductors the error is related to the deviation in the superconductor cut from an *isotherm* of the exact solution.

Our final result concerning insulators and superconductors, corresponding to Elrod's 'Theorem II' [5], can thus be written as

$$q^I \leqslant q \leqslant q^S \tag{19}$$

allowing for estimation of both upper and lower bounds for conduction heat transfer.

One-dimensional estimates

Upper bounds. We briefly describe here some commonly-used *one-dimensional* estimates for heat transfer that can be interpreted in terms of the proofs given above. Consider the problem of conduction in the wedge shown in Fig. 4, governed by the differential and variational forms given in equations (1) and (9), respectively. If we make the assumption of one-dimensionality, $\theta = \theta(x)$, and require that the heat transfer rate be the same at any cross-section (x = constant), we arrive at the usual equation for (approximate) one-dimensional heat transfer

$$\mathrm{d}/\mathrm{d}x[A(x)\,\mathrm{d}\theta(x)/\mathrm{d}x] = 0, \quad \theta(0) = 1, \quad \theta(1) = 0 \tag{20a}$$

where $A(x)$ is the area of the wedge per unit depth, $A(x) = 1+x$. From equation (20a) we can readily find the solution, θ

$$\theta(x) = \int_x^1 1/A(\xi)\,\mathrm{d}\xi \Big/ \int_0^1 1/A(\xi)\,\mathrm{d}\xi \tag{20b}$$

and corresponding heat transfer rate $q^{1\mathrm{D}}$ ($= -A(x)\,\mathrm{d}\theta/\mathrm{d}x$)

$$q^{1\mathrm{D}} = 1 \Big/ \int_0^1 1/A(\xi)\,\mathrm{d}\xi. \tag{20c}$$

It is then straightforward to show that the variational statement associated with equation (20) is given by

$$q^{1\mathrm{D}} = \inf_{v \in \mathscr{B}^{1\mathrm{D}}} \int_0^1 (\mathrm{d}v/\mathrm{d}\xi)^2 A(\xi)\,\mathrm{d}\xi \tag{21a}$$

where $\mathscr{B}^{1\mathrm{D}}$ is the space of one-dimensional functions $v(x)$ that satisfy the essential boundary conditions, $v(0) = 1$, $v(1) = 0$. Using the fact that $v = v(x)$ only, equation (21a) can be rewritten as

$$q^{1\mathrm{D}} = \inf_{v \in \mathscr{B}^{1\mathrm{D}}} \langle (\nabla v)^2 \rangle. \tag{21b}$$

Now, since $\mathscr{B}^{1\mathrm{D}} \subset \mathscr{B}$, it follows from the arguments of the previous section, expressions (10) and (11), that $q^{1\mathrm{D}}$ is an upper bound for the actual heat transfer rate

$$q \leqslant q^{1\mathrm{D}}. \tag{22}$$

This reflects the fact that the one-dimensional assumption is equivalent to replacing all resistances in the y-direction with superconductors. Note the one-dimensional estimations are most useful (i.e. accurate) when the insulated boundaries have small slope, as in this situation the isotherms are, in fact, close to vertical lines.

Lastly, we show how to obtain a 'simplest' upper bound to one-dimensional (and thus, from inequality (22), two-dimensional) conduction heat transfer. We consider the effect of *area averaging*, corresponding to replacement of the original domain with a constant cross-section body possessing the same area and length as the original domain. This problem is trivial to solve, giving

$$q^{1\mathrm{D},A} = \int_0^1 A(\xi)\,\mathrm{d}\xi. \tag{23}$$

In order to show that $q^{1\mathrm{D},A}$ is larger than $q^{1\mathrm{D}}$, we make use of the following Schwarz inequality for a function $f(x) > 0$

$$1 = \left(\int_0^1 f^{1/2} f^{-1/2}\,\mathrm{d}x\right)^2 \leqslant \left(\int_0^1 f\,\mathrm{d}x\right)\left(\int_0^1 f^{-1}\,\mathrm{d}x\right). \tag{24a}$$

Identifying $f(x)$ as $A(x)$, we find that

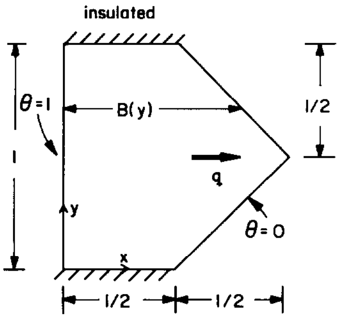

FIG. 5. Geometry used to illustrate the one-dimensional estimation techniques for lower bounds.

$$q^{1D,A}/q^{1D} = \int_0^1 1/A(\xi)\,d\xi \int_0^1 A(\xi)\,d\xi \geqslant 1 \quad (24b)$$

which gives the following inequality:

$$q \leqslant q^{1D} \leqslant q^{1D,A}. \quad (25)$$

The effect of area averaging is to provide an upper bound that is greater yet than the one-dimensional approximation. Intuitively, $q^{1D,A} > q^{1D}$ due to the fact that the decrease in resistance at the small-area bottlenecks is proportionately greater than the increase in resistance in the large-area regions of the wedge. If we evaluate our estimates in inequality (25) for the wedge shown in Fig. 4, we find that $q^{1D,A} = 3/2 = 1.50$ and $q^{1D} = 1/\ln 2 = 1.44$. The 'exact', two-dimensional solution is found (numerically) to be $q = 1.38$ [8].

Lower bounds. By analogy with the previous section, we demonstrate here the use of one-dimensional estimates to arrive at lower bounds for heat transfer. Consider the two-dimensional conduction problem given in Fig. 5, governed by the differential and variational forms given in equations (1) and (9), respectively. If we now neglect the heat flux in the y-direction at any point inside D, we arrive at another form of one-dimensional equation in x, where y now appears as a parameter

$$\frac{d^2\theta}{dx^2} = 0, \quad \theta|_{x=0} = 1, \quad \theta|_{x=B(y)} = 0 \quad (26a)$$

with the solution $\theta(x)$ and heat transfer q^{1d} obtained as

$$\theta = 1 - x/B(y) \quad (26b)$$

$$q^{1d} = \int_0^1 \frac{d\xi}{B(\xi)}. \quad (26c)$$

The variational formulation associated with equations (26) can be written as

$$q^{1d} = \inf_{v \in B^{1d}} \int_0^1 \int_0^{B(y)} \left[\frac{\partial v(x,y)}{\partial x}\right]^2 dx\,dy \quad (27)$$

where B^{1d} is the space of functions of two variables $v(x, y)$ which are continuous and differentiable with respect to the x variable, satisfy boundary conditions from equations (26a), but might be discontinuous in the y-direction. The equivalence of equation (27) to equations (26) can be shown using standard variational techniques.

In order to arrive at the required lower bounds we define $v_0(x, y)$ as the exact solution of the two-dimensional problem (1). Then, recognizing that $v_0(x, y) \in B^{1d}$ and using equation (27) we obtain

$$q^{1d} \leqslant \int_0^1 \int_0^{B(y)} (\partial v_0/\partial x)^2\,dx\,dy$$
$$\leqslant \int_0^1 \int_0^{B(y)} [(\partial v_0/\partial x)^2 + (\partial v_0/\partial y)^2]\,dx\,dy = q. \quad (28)$$

The lower bound reflects the fact that this form of one-dimensional estimate is equivalent to replacing all resistances in the x-direction with insulators.

Now again, as in the previous section, we consider the effect of area averaging in order to provide a 'simplest' lower bound for q^{1d} and thus for q. We replace now the original domain given on Fig. 5 by the region with constant length

$$\bar{B} = \int_0^1 B(\xi)\,d\xi$$

possessing the same area and height as the original domain. The heat conduction problem in the 'averaged' domain is trivially solved

$$q^{1d,B} = 1\Big/\int_0^1 B(\xi)\,d\xi. \quad (29a)$$

Identifying $f(x)$ from equation (24a) with $B(y)$ and using equation (26c) we arrive at

$$q^{1d,B} \leqslant q^{1d} \leqslant q. \quad (29b)$$

Intuitively, we can understand inequality (29b) by noting that the thermal resistance is directly proportional to $B(y)$, and that the increases in $B(y)$ (resistance) for $B(y) < \bar{B}$ are proportionately greater than the decreases in $B(y)$ (resistance) for $B(y) > \bar{B}$, thus giving $q^{1d,B} < q^{1d}$. As an example we choose the geometry of Fig. 5, $B(y) = 1 - |y - 0.5|$, $0 \leqslant y \leqslant 1$, for which we obtain $q^{1d,B} = 4/3 = 1.333 < q^{1d} = 2\ln 2 = 1.386 < q = 1.565$ [8].

Use of one-dimensional estimates of the kind described here is quite widespread. However, it is generally not stated whether such estimates constitute lower or upper bounds, or what the hierarchy is in terms of degree of approximation. The simple proofs given here should help in this regard.

Extension to convective boundary conditions

We have considered so far the case with two isothermal boundaries ∂D_1 and ∂D_0. However, our results readily extend to the case where these boundaries are exposed to the ambient temperatures (T_1 and T_0, respectively) through a convective heat transfer coefficient, $h(\mathbf{x}) > 0$. Our problem statement (1) then becomes

$$\nabla^2\theta = 0 \qquad \text{in } D \qquad (30a)$$

$$\nabla\theta\cdot\mathbf{n} = 0 \qquad \text{on } \partial D_2 \qquad (30b)$$

$$\nabla\theta\cdot\mathbf{n} = -h'(\theta-1) \qquad \text{on } \partial D_1$$

$$\nabla\theta\cdot\mathbf{n} = -h'\theta \qquad \text{on } \partial D_0 \qquad (30c)$$

where h' is a non-dimensional heat transfer coefficient (i.e. Biot number).

Following a procedure analogous to that used to arrive at equation (9), it can be shown that the non-dimensional heat transfer rate associated with equations (30) is given by

$$q = \inf_{v\in\tilde{\mathscr{B}}} \tilde{I}(v) \qquad (31a)$$

where $\tilde{I}$ is the 'modified' Dirichlet functional

$$\tilde{I}(v) = \langle(\nabla v)^2\rangle + \int_{\partial D_1} h'(s)(v-1)^2\,\mathrm{d}s + \int_{\partial D_0} h'(s)v^2\,\mathrm{d}s \qquad (31b)$$

and $\tilde{\mathscr{B}} \equiv \mathscr{H}^1$, as all boundary conditions in equations (30) are natural. The fact that we can once again express the heat transfer rate as the minimum of a functional implies that the (variational) results for the constant temperature case directly extend to the case of convective boundary conditions treated in ref. [5].

PART 2. AN UPPER BOUND FOR FORCED CONVECTION HEAT TRANSFER IN 'TURBULENT' PLANE COUETTE FLOW

We are concerned here with an upper bound for heat transport by forced convection in 'turbulent' plane Couette flow. Closely related problems have been considered previously by Howard [2, 9] and Busse [10, 11]. In particular, Howard obtained an upper bound for heat transport by *natural* convection between two infinite horizontal plates for a given Rayleigh number [9], and for the dissipation function for plane Couette flow for a given Reynolds number [2]. Busse [10, 11] subsequently improved on some of Howard's results using the technique of multiple boundary layers.

As is well known, the variational principle for conduction heat transfer utilized in Part 1 of this paper is no longer relevant for the convection problem, and thus another method must be used to obtain the required estimations of heat transfer. Howard's approach entails the construction of exact integral expressions ('power constraints') related to global properties of the flow. One then looks for a supremum of the quantities of interest (e.g. heat transport) subject to these energy integral constraints, the boundary conditions of the problem, and perhaps incompressibility. Integral inequalities serve as first estimates for upper bounds, however, more refined results can be obtained by direct investigation of the appropriate Euler equations.

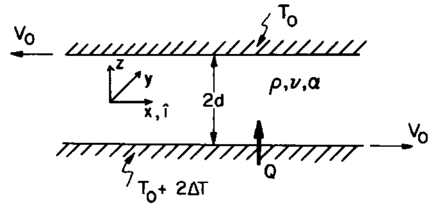

FIG. 6. Geometry definition for forced convection in plane Couette flow between infinite parallel plates.

This paper can be considered as an extension of Howard's ideas to the case of forced convection. In particular, heat transport in the plane Couette geometry is estimated in terms of the Peclet number and the (given) viscous dissipation (e.g. momentum transport) of the flow. It is clear that the momentum equations are independent of the heat equation for the forced convection case, and thus the momentum transport can be either independently estimated, or taken from experimental data. We use here only simple integral inequalities (almost entirely taken from Howard's papers), with no attempt made to obtain the extremizing fields. It is therefore clear that the result presented is not optimal, and can be improved to provide closer agreement with experiment.

The results obtained in this part of the paper can also be related to Reynolds' conjecture [12] concerning the similarity between momentum and heat transfer, now known as the Reynolds analogy [6]. The simplicity and practical ramifications of this idea are extremely intriguing, however, to date, it has been rigorously demonstrated only for the flat plate laminar boundary layer. We show here that the heat flux is, indeed, bounded by the momentum flux for a non-trivial class of flows, an indication that the Reynolds analogy can perhaps be systematically extended to more complicated situations.

Problem statement

We consider incompressible flow and heat transport between two infinite isothermal rigid plates separated by a distance $2d$, as shown in Fig. 6. The plates are moving in opposite directions with velocities $\mp V_0$, with the top plate at temperature $T = T_0$ and the bottom plate at temperature $T = T_0+2\Delta T$. Using d for the length scaling, d/V_0 for time, V_0 for velocity, ρV_0^2 for pressure, and ΔT for temperature, the non-dimensional equations for forced convection are given by

$$\partial\mathbf{V}/\partial t+\mathbf{V}\cdot\nabla\mathbf{V} = -\nabla P+1/Re\,\nabla^2\mathbf{V};$$

$$\nabla\cdot\mathbf{V} = 0; \quad \mathbf{V} = \mp\mathbf{i} \text{ on } z = \pm 1 \qquad (32a)$$

$$\partial\Theta/\partial t+\mathbf{V}\cdot\nabla\Theta = 1/Pe\,\nabla^2\Theta; \quad \Theta = \mp 1 \text{ on } z = \pm 1 \qquad (32b)$$

where $Re = V_0 d/\nu$ is the Reynolds number, $Pe =$

$Re\,Pr$ the Peclet number, $Pr = \nu/\alpha$ the Prandtl number, and $\Theta = [(T-T_0)/\Delta T]-1$ the non-dimensional temperature. Here ν is the kinematic viscosity and α the thermal diffusivity of the fluid. It should be noted that we have neglected the (forcing) viscous dissipation term in thermal energy equation (32b), due to its minimal influence on the temperature field for the parameter range of interest here. However, the viscous dissipation (i.e. wall shear stress) will nevertheless play a critical role in determining the heat transfer, in that by specifying a *fixed* dissipation we impose a constraint on the possible velocity fields that enter into the $\mathbf{V}\cdot\nabla\Theta$ convective term of the energy equation.

Following Howard [2, 9], we assume that for any function $\mathbf{V}$, P, Θ averages over the entire domain (defined here as $\langle\cdot\rangle$) exist, and are independent of time. Using this assumption, the energy integral of equation (32b) (multiplication by Θ followed by averaging over the domain) results in effectively the same expression for wall heat transfer as for the conduction problem studied in Part 1. However, although the heat transfer rate for the conduction and convection problems has formally the same expression in terms of the Dirichlet integral, $\bar{Q} = k\Delta T\langle(\nabla\Theta)^2\rangle/d$ (here $\bar{Q}$ is the average heat flux at the wall), the variational principle utilized in Part 1 is no longer relevant, as the convection heat transfer problem has no 'extremum' properties. Some new approach must therefore be found in searching for estimations of $\bar{Q}$.

In order to define this approach we utilize the technique of space averaging. We assume that for any function $\mathbf{V}$, P, Θ horizontal averages (over planes z = constant) exist, and are independent of time (see the Appendix). All quantities are then split into 'mean' and 'fluctuating' parts. The horizontal averages will be denoted by an overbar, and represent the mean values of the given quantities

$$\mathbf{V} = u_0(z)\mathbf{i}+\mathbf{v}; \quad P = p_0(z)+p; \quad \Theta = \theta_0(z)+\theta \tag{33a}$$

$$\bar{\mathbf{v}} = \bar{p} = \bar{\theta} = 0. \tag{33b}$$

Averaging the momentum and temperature equations we obtain the following expressions for the mean fields:

$$u_0(z) = -z+Re\left[\int_{-1}^{z}\overline{uw}\,\mathrm{d}\zeta-(1+z)\langle uw\rangle\right] \tag{34a}$$

$$\theta_0(z) = -z+Pe\left[\int_{-1}^{z}\overline{\theta w}\,\mathrm{d}\zeta-(1+z)\langle \theta w\rangle\right] \tag{34b}$$

where averaging over the entire domain can be defined in terms of horizontal averages as

$$\langle f\rangle = 1/2\int_{-1}^{1}\bar{f}\,\mathrm{d}z.$$

Now defining $\bar{F}$ as the mean momentum flux (i.e. shear stress) and $\bar{Q}$ as the mean heat flux, we obtain from equations (34)

$$f = \bar{F}/(\mu V_0/d) = -\mathrm{d}u_0/\mathrm{d}z|_{z=\pm 1} = 1+Re\langle uw\rangle \tag{35a}$$

$$q = \bar{Q}/(k\Delta T/d) = -\mathrm{d}\theta_0/\mathrm{d}z|_{z=\pm 1} = 1+Pe\langle \theta w\rangle \tag{35b}$$

where μ is the viscosity and k the thermal conductivity of the fluid. The parameter f defined in equation (35a) represents both the momentum flux (wall shear stress) and the viscous dissipation, while q represents the heat flux (or Nusselt number).

Multiplying the perturbation momentum equation by $\mathbf{v}$ and the perturbation heat equation by θ and integrating over the domain, the two 'energy' integrals are obtained

$$\langle uw\rangle = 1/Re\langle(\nabla\mathbf{v})^2\rangle+Re\langle(\overline{uw}-\langle uw\rangle)^2\rangle \tag{36a}$$

$$\langle \theta w\rangle = 1/Pe\langle(\nabla\theta)^2\rangle+Pe\langle(\overline{\theta w}-\langle \theta w\rangle)^2\rangle \tag{36b}$$

where we have used homogeneity and the mean-field expressions given by equations (34). Equations (34)–(36) appear exactly as in the earlier work by Howard [2, 9].

In this paper we are looking for relatively simple estimations for heat transfer. For the conduction problem described in Part 1 we could utilize a variational principle to reduce the problem to tractable form. Unfortunately, for turbulent convection no variational statement is known to exist. However, we can maintain the integral nature of the analysis of Part 1 by using power integrals (36) to construct 'artificial' functionals that can then be used to *bound* q (although the actual q will not correspond to this extremum).

We are therefore searching here for an upper bound for q, the heat flux, in terms of the Reynolds and Prandtl numbers for a given value of momentum flux, f. The problem can be stated as follows.

Find an upper bound for $q-1 = Pe\langle\theta w\rangle$ when the functions θ, $\mathbf{v}$ satisfy the following constraints: $\theta = \mathbf{v} = 0$ on the boundaries; the power integrals given by equations (36); $f-1 = Re\langle uw\rangle$; $\nabla\cdot\mathbf{v} = 0$. Here f can be considered as given.

The form of equations (34a) and (34b), (35a) and (35b), and (36a) and (36b) suggests a strong similarity in the heat and momentum transport mechanisms. Thus we expect that the heat flux can be bounded in terms of the momentum flux, which corresponds to a (somewhat weak) statement of the classical Reynolds analogy.

An upper bound for heat transport

In this section, we obtain an upper bound for (turbulent) heat transport. First, it is well known that if the value of the Reynolds number is sufficiently small, the unique stationary solution of equations (32) corresponds to linear plane Couette flow ($f = 1$), with purely conducting heat transfer ($q = 1$, a lower bound for convection heat transfer for any value of the Reynolds number). However, with an increase of the Reynolds number, the linear velocity profile becomes unstable, and as from equations (36) we see that both $\langle uw \rangle$ and $\langle \theta w \rangle$ are positive, f and q must be greater than unity for any new kind of stationary flow.

To determine how large $\langle \theta w \rangle$ (and hence, q) can be, we use equation (36b) to rewrite equation (35b) as a homogeneous functional

$$q-1 = \frac{1-1/Pe\langle(\nabla\theta)^2\rangle/\langle\theta w\rangle}{\langle(1-\overline{\theta w}/\langle\theta w\rangle)^2\rangle} \tag{37}$$

and search for a bound of this quantity. We see that the functional is homogeneous, and thus an increase in the 'amplitudes' of the functions (θ, w) does not endanger the possibility of an upper bound for q. Functional (37) cannot be bounded from above only if the denominator can become arbitrarily small. From the Schwarz inequality ($\langle f \rangle^2 \leqslant \langle f^2 \rangle$), and the boundary conditions of the flow, we easily obtain that the denominator equals zero only for the trivial case $\overline{\theta w} \equiv 0$ (i.e. $q = 1$). However, if $\overline{\theta w}$ is a constant almost everywhere and goes to zero only near the walls (which is consistent with the 'real' boundary layer structure), the value of the denominator will become quite small. It remains to be shown that the denominator can in fact be bounded from below so that q is bounded from above. In essence, this addresses the question of whether by fixing the dissipation we sufficiently restrict the range of excited scales that enter into the thermal transport process.

Slightly reformulating Howard's estimations for terms similar to that appearing in the denominator of equation (37), we obtain

$$\langle(1-\overline{\theta w}/\langle\theta w\rangle)^2\rangle \geqslant 1/6\langle\theta w\rangle/[\langle(\nabla\theta)^2\rangle\langle(\nabla\mathbf{v})^2\rangle]^{1/2}. \tag{38}$$

A complete derivation of this inequality is given in the Appendix. Now, using equations (37) and (38) we obtain

$$q-1 \leqslant 6(1-x/Pe)\sqrt{x}\left[\frac{\langle(\nabla\mathbf{v})^2\rangle}{\langle\theta w\rangle}\right]^{1/2} \tag{39}$$

where $x = \langle(\nabla\theta)^2\rangle/\langle\theta w\rangle$. For any x, $(1-x/Pe)\sqrt{x} \leqslant 2(Pe/27)^{1/2}$, and thus

$$q-1 \leqslant \left[\frac{16Pe^2}{3}\frac{\langle(\nabla\mathbf{v})^2\rangle}{Re\langle uw\rangle}\frac{Re\langle uw\rangle}{Pe\langle\theta w\rangle}\right]^{1/2}. \tag{40}$$

From equation (36a) we have that $\langle(\nabla\mathbf{v})^2\rangle \leqslant Re\langle uw\rangle$, and thus, using equations (35), we obtain

$$q-1 \leqslant 4Pe\sqrt{(f-1)}/\sqrt{(3(q-1))} \tag{41a}$$

or

$$q-1 \leqslant (16/3)^{1/3}Pe^{2/3}(f-1)^{1/3} \tag{41b}$$

which is the required bound for q in terms of Pe and f.

The value of f represents the non-dimensional momentum flux through the boundaries, and thus the bound given in equations (41) can be considered as theoretical support to Reynolds' proposed analogy between heat and momentum transfer. Although there is not, to our knowledge, any experimental data for heat transport in turbulent Couette flow, comparison of equation (41b) with heuristic models [13] indicates that equation (41b) probably overestimates the heat transfer by several orders of magnitude. Furthermore, the functional dependence of q on f is not that expected from Reynolds' analogy. These discrepancies may be due to the integral estimates inherent in equation (41b) (i.e. equations (38)–(40)), or the fact that the low-order moments given in equation (36b) do not sufficiently restrict the class of admissible functions. The merit of bound (41b) is clearly not in its predictive value, but rather in the support it lends to the concept that the mechanisms of momentum and heat transport in turbulent flow are intimately linked.

Acknowledgements—This work was supported by a Bantrell Postdoctoral Fellowship (M.M.) and a Rockwell Assistant Professorship (A.T.P.).

REFERENCES

1. B. A. Finlayson, *The Method of Mean Weighted Residuals and Variational Principles*. Academic Press, New York (1972).
2. L. N. Howard, Bounds on flow quantities, *Ann. Rev. Fluid Mech.* **4,** 473–494 (1973).
3. G. Strang and G. J. Fix, *An Analysis of the Finite Element Method*. Prentice-Hall, Englewood Cliffs, New Jersey (1973).
4. S. G. Mikhlin, *The Problem of the Minimum of a Quadratic Functional*. Holden-Day, San Francisco (1965).
5. H. G. Elrod, Two simple theorems for establishing bounds on the total heat flow in steady-conduction problems with convective boundary conditions, *Trans. ASME J. Heat Transfer* 65–70 (February 1974).
6. W. H. Kays and M. E. Crawford, *Convective Heat and Mass Transfer*. McGraw-Hill, New York (1980).
7. M. Magen, A new kind of linear elliptic boundary condition: application to hydrodynamics, *SIAM J. Appl. Math.*, to be published.
8. *NEKTON User's Manual*. Nektonics Inc., Bedford, Massachusetts (1986).
9. L. N. Howard, Heat transport by turbulent convection, *J. Fluid Mech.* **17,** 405–432 (1963).
10. F. H. Busse, On Howard's upper bound for heat transport by turbulent convection, *J. Fluid Mech.* **37,** 457–477 (1969).
11. F. H. Busse, Bounds for turbulent shear flow, *J. Fluid Mech.* **41,** 219–240 (1970).
12. O. Reynolds, On the extent and action of the heating surface for steam boilers, *Proc. Manchr Lit. Phil. Soc.* **14,** 7–12 (1874).
13. A. A. Sonin, Private communication (1986).
14. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press, Oxford (1961).

APPENDIX

A complete derivation of inequality (38) is presented here. Although these derivations are almost entirely taken from the work of Howard [2, 9], we believe that it is worthwhile to summarize them here because of their importance in understanding our final result.

First, we give the formal definition of horizontal averaging. Define in the horizontal plane x, y a region B with area

$$A = \int_B \mathrm{d}B$$

(e.g. if B is defined by $|x| \leqslant a, |y| \leqslant b$, then $A = 4ab$). For a given function $F(x, y, z)$ we consider the following limit:

$$\bar{F}(z) = \lim_{B\to\infty}\left[\frac{1}{A}\int_B F(x,y,z)\,\mathrm{d}B\right]. \tag{A1}$$

By definition, if the limit on the right-hand side exists and does not depend on the way B goes to infinity, then this limit (depending on z only) represents the horizontal average of the function $F(x, y, z)$.

Consider now a function $f(z)$ continuous on the interval $(-1, 1)$ with piecewise continuous derivative f_z. Then, if $f(\pm 1) = 0$ and if

$$\int_{-1}^{1} f_z^2\,\mathrm{d}z$$

exists, then

$$f^2(z) \leqslant 2(1-z)\langle f_z^2\rangle$$

and also

$$f^2(z) \leqslant 2(1+z)\langle f_z^2\rangle.$$

To show this, we use the Schwarz inequality

$$f(z)^2 = \left[\int_{-1}^{z} f_\zeta(\zeta)\,\mathrm{d}\zeta\right]^2 \leqslant \int_{-1}^{z}\mathrm{d}\zeta\int_{-1}^{z} f_\zeta^2(\zeta)\,\mathrm{d}\zeta \leqslant 2(1+z)\langle f_z^2\rangle$$

$$f(z)^2 = \left[\int_{z}^{1} f_\zeta(\zeta)\,\mathrm{d}\zeta\right]^2 \leqslant \int_{z}^{1}\mathrm{d}\zeta\int_{z}^{1} f_\zeta^2(\zeta)\,\mathrm{d}\zeta \leqslant 2(1-z)\langle f_z^2\rangle. \tag{A2}$$

Now, for any function $u(x, y, z)$ which satisfies the condition $u(x, y, \pm 1) = 0$, we obtain

$$\overline{u^2} = \lim_{B\to\infty}\frac{1}{A}\int_B u^2\,\mathrm{d}B \leqslant 2(1-z)\lim_{B\to\infty}\frac{1}{A}\int_B\frac{1}{2}\int_{-1}^{1}[u(x,y,z)_z]^2\,\mathrm{d}z\,\mathrm{d}B$$
$$= 2(1-z)\langle(u_z)^2\rangle \leqslant 2(1-z)\langle(\nabla u)^2\rangle. \tag{A3}$$

We can obtain a similar result integrating from the other wall, giving

$$\overline{u^2} \leqslant 2(1+z)\langle(\nabla u)^2\rangle \;\text{good for } z \text{ close to } -1;\quad \overline{u^2} \leqslant 2(1-z)\langle(\nabla u)^2\rangle \;\text{good for } z \text{ close to } 1. \tag{A4}$$

Then, for θ and w representing the quantities defined in Part 2 of this paper, we use the Schwarz inequality and the above results to obtain

$$|\overline{\theta w}|^2 = \left|\lim_{B\to\infty}\frac{1}{A}\int_B \theta w\,\mathrm{d}B\right|^2 \leqslant \left[\lim_{B\to\infty}\frac{1}{A}\int_B|\theta w|\,\mathrm{d}B\right]^2$$
$$\leqslant \left[\lim_{B\to\infty}\frac{1}{A}\int_B\theta^2\,\mathrm{d}B\right]\left[\lim_{B\to\infty}\frac{1}{A}\int_B w^2\,\mathrm{d}B\right]$$
$$= \overline{\theta^2}\,\overline{w^2} \leqslant 4(1-z)^2\langle(\nabla\theta)^2\rangle\langle(\nabla w)^2\rangle. \tag{A5}$$

Expanding the gradient to include all the velocity components, we find that

$$|\overline{\theta w}| \leqslant 2(1-z)[\langle(\nabla\theta)^2\rangle\langle(\nabla\mathbf{v})^2\rangle]^{1/2}. \tag{A6}$$

From equation (36b) we see that $\langle\theta w\rangle$ is positive, and thus we can rewrite the last inequality as

$$|\overline{\theta w}| \leqslant (1-z)\langle\theta w\rangle/\delta \tag{A7}$$

where

$$2\delta = \left[\frac{\langle\theta w\rangle^2}{\langle(\nabla\theta)^2\rangle\langle(\nabla\mathbf{v})^2\rangle}\right]^{1/2} \leqslant \frac{1}{\sqrt{Ra}} \simeq \frac{2}{\sqrt{1708}}. \tag{A8}$$

This last inequality is borrowed from the Benard problem [14], where it is proven that equation (A8) holds for any θ and $\mathbf{v}$ satisfying $\theta = \mathbf{v} = 0$ on the boundaries and $\nabla\cdot\mathbf{v} = 0$. For the case of *natural* convection Ra has the meaning of the critical Rayleigh number, but in our case of *forced* convection this value can be considered as simply the minimum eigenvalue of an appropriately-defined eigenproblem. For the following derivation it is only important that δ be less than unity.

We can thus obtain

$$|\overline{\theta w}| \leqslant (1+z)\langle\theta w\rangle/\delta;\quad |\overline{\theta w}| \leqslant (1-z)\langle\theta w\rangle/\delta$$
$$1-|\overline{\theta w}|/\langle\theta w\rangle \geqslant 1-(1+z)/\delta \;\text{good for } z \text{ close to } -1;\quad 1-|\overline{\theta w}|/\langle\theta w\rangle \geqslant 1-(1-z)/\delta \;\text{good for } z \text{ close to } 1. \tag{A9}$$

For the intervals $(-1, -1+\delta)$ and $(1-\delta, 1)$ we see that both sides of inequalities (A9) are positive, and thus

$$[1-|\overline{\theta w}|/\langle\theta w\rangle]^2 \geqslant \left[1-\frac{1+z}{\delta}\right]^2 \;\text{for } z\in(-1,-1+\delta);$$
$$[1-|\overline{\theta w}|/\langle\theta w\rangle]^2 \geqslant \left[1-\frac{1-z}{\delta}\right]^2 \;\text{for } z\in(1-\delta,1). \tag{A10}$$

On the basis of this result we then readily derive (recall $\delta < 1$)

$$\langle(1-\overline{\theta w}/\langle\theta w\rangle)^2\rangle = \frac{1}{2}\int_{-1}^{1}(1-\overline{\theta w}/\langle\theta w\rangle)^2\,\mathrm{d}z$$
$$\geqslant \frac{1}{2}\int_{-1}^{1}(1-|\overline{\theta w}|/\langle\theta w\rangle)^2\,\mathrm{d}z$$
$$\geqslant \frac{1}{2}\int_{-1}^{-1+\delta}(1-|\overline{\theta w}|/\langle\theta w\rangle)^2\,\mathrm{d}z + \frac{1}{2}\int_{1-\delta}^{1}(1-|\overline{\theta w}|/\langle\theta w\rangle)^2\,\mathrm{d}z$$
$$\geqslant \frac{1}{2}\int_{-1}^{-1+\delta}\left(1-\frac{1+z}{\delta}\right)^2\mathrm{d}z + \frac{1}{2}\int_{1-\delta}^{1}\left(1-\frac{1-z}{\delta}\right)^2\mathrm{d}z = \frac{\delta}{3}. \tag{A11}$$

Substituting expressions (A8) in expressions (A11) we eventually obtain

$$\langle(1-\overline{\theta w}/\langle\theta w\rangle)^2\rangle \geqslant 1/6\langle\theta w\rangle/[\langle(\nabla\theta)^2\rangle\langle(\nabla\mathbf{v})^2\rangle]^{1/2} \tag{A12}$$

which is inequality (38) appearing in Part 2 of this paper. It should be noted that for a turbulent flow we expect the estimate (A12) of $\langle(1-\overline{\theta w}/\langle\theta w\rangle)^2\rangle$ to be relatively sharp, as the neglected portion of the integral in expression (A11) corresponds to the well-mixed 'core' in which $\overline{\theta w} \simeq \langle\theta w\rangle$.

LIMITES ENTRE LA CONDUCTION THERMIQUE ET LA CONVECTION FORCEE

Résumé—On considère les frontières de transfert de chaleur dans quelques problèmes classiques de conduction et de convection forcée de la chaleur. Les limites de conduction sont basées sur une formulation variationnelle dans laquelle un enrichissement ou une restriction de l'espace des fonctions à minimiser conduit à une frontière inférieure ou supérieure. On montre que l'addition de "coupes" isolantes crée un élargissement de l'espace et par suite d'une sous-estimation du transport, tandis que l'addition de coupes supraconductrices conduit à une contraction de l'espace et à une surestimation. Ces résultats constituent une preuve formelle de quelques théorèmes proposés par Elrod (*Trans. ASME J. Heat Transfer* 65–70 (1974)). On présente aussi une limite supérieure de la convection forcée dans l'écoulement turbulent de Couette à partir de la méthode des moyennes horizontales et des intégrales puissance. En particulier, on montre que pour un transfert de quantité de mouvement donné, le flux thermique à la paroi peut être limité en dessous par une fonction des nombres de Reynolds et de Prandtl. Cette relation entre la tension de frottement et le flux de chaleur sert de support théorique à l'analogie de Reynolds entre les transports de quantité de mouvement et de chaleur pour les écoulements turbulents sans séparation.

GRENZEN FÜR DEN WÄRMETRANSPORT DURCH LEITUNG UND ERZWUNGENE KONVEKTION

Zusammenfassung—Es werden Grenzen für den Wärmetransport bei verschiedenen klassischen Problemen der Wärmeleitung und der erzwungenen Konvektion entwickelt. Die Grenzen der Wärmeleitung beruhen auf einem Variations-Verfahren, bei welchem eine systematische Erweiterung und Einschränkung des Raumes der Minimierungsfunktionen zu unteren und oberen Grenzen führt. Es wird gezeigt, daß das Hinzufügen einer Isolator-Scheibe zu einer Erweiterung des Raumes führt, wodurch der Transport als zu klein berechnet wird. Gerade umgekehrt ist die Wirkung beim Hinzufügen eines Supraleiters. Unsere Ergebnisse begründen einen formalen Beweis der verschiedenen Theoreme, die von Elrod vorgestellt worden sind. Eine obere Grenze für den Wärmetransport durch erzwungene Konvektion in einer turbulenten ebenen Couette-Strömung wird vorgestellt. Im einzelnen wird gezeigt, daß für einen festen vorgegebenen Impulstransport der Wärmetransport von der Platte als eine Funktion der Reynolds- und Prandtl-Zahlen nach oben abgegrenzt werden kann. Diese Beziehung zwischen der Schubspannung und der Wärmestromdichte dient als theoretische Unterstützung der Reynolds-Analogie für turbulente, nicht abgelöste Strömung.

ГРАНИЦЫ МЕЖДУ КОНДУКТИВНЫМ И ВЫНУЖДЕННОКОНВЕКТИВНЫМ ТЕПЛООБМЕНОМ

Аннотация—Определены границы связей переноса тепла в различных классических задачах кондуктивного и вынужденноконвективного теплообмена. Для теплопроводности связи основаны на вариационной постановке, где систематическое расширение и ограничение пространства величин, переходящих в минимизирующие функции, приводит к нижним и верхним связям, соответственно. Показано, что добавление непроводящих 'отрезков' дает расширение пространства и, соответственно, занженные данные по переносу, в то время как добавление сверхпроводящих отрезков приводит к сжатию пространства и завышенным результатам. Наши данные формально вытекают из нескольких теорем Эльрода (*Trans. ASME J. Heat Transfer* 65–70 (February 1974)). Получена верхняя связь для вынужденноконвективного теплопереноса в турбулентном плоском течении Куэтта, основанном на методе горизонтальных усреднений и интегралов энергии. В частности, показано, что для заданного переноса импульса интенсивноств теплопереноса от стенки может быть ограничена сверху как функция чисел Рейнольдса и Прандтля. Это соотношение между касательным напряжением и тепловым потоком служит теоретическим педтверждением рейнольдсовской аналогии переноса импульса/тепла для турбулентного неоторвавшегося потока.